Logical Properties of Nonmonotonic Causal Theories and the Action Language $\mathcal{C}+$

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Abstract

The formalism of nonmonotonic causal theories (Giunchiglia, Lee, Lifschitz, McCain, Turner, 2004) provides a general-purpose formalism for nonmonotonic reasoning and knowledge representation, as well as a higher level, special-purpose notation, the action language $\mathcal{C}+$, for specifying and reasoning about the effects of actions and the persistence ('inertia') of facts over time. In this paper we investigate some logical properties of these formalisms. There are two motivations. From the technical point of view, we seek to gain additional insights into the properties of the languages when viewed as a species of conditional logic. From the practical point of view, we are seeking to find conditions under which two different causal theories, or two different action descriptions in $\mathcal{C}+$, can be said to be equivalent, with the further aim of helping to decide between alternative formulations when constructing practical applications.

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1 Introduction

The formalism of nonmonotonic causal theories, presented by Giunchiglia, Lee, Lifschitz, McCain and Turner [GLL$^+$04], is a general-purpose language for knowledge representation and nonmonotonic reasoning. A *causal theory* is a set of causal rules each of which is an expression of the form

$$F \Leftarrow G$$

where F and G are formulas of an underlying propositional language and $F \Leftarrow G$ corresponds to the statement 'if G, then F has a cause' (which is not the same as saying that G is a cause for F).

Associated with causal theories is the action language $\mathcal{C}+$, also presented in [GLL$^+$04]. This may be viewed as a higher-level formalism for defining classes of causal theories in a concise and natural way, for the purposes of specifying and reasoning about the effects of actions and the persistence, or 'inertia', of facts through time, with support for indirect effects, non-deterministic actions and concurrency. The two closely-related formalisms have been used to represent standard domains from the knowledge representation literature.

In this paper, we investigate some logical properties of these formalisms. There are two motivations. The first is technical, to gain new insights into the languages when they are viewed as species of conditional logic. For example, Turner [Tur99] presents a more general formalism called the 'logic of universal causation'. A rule $F \Leftarrow G$ of a causal theory can be expressed equivalently in this logic by the formula

$$G \rightarrow \mathbf{C}\, F$$

where $\mathbf{C}$ is a modal operator standing for 'there is a cause for' (and $\rightarrow$ is truth-functional, 'material' implication). Since $\mathbf{C}$ is a normal modal operator whose logic is at least as strong as $S5$, some logical properties of $F \Leftarrow G$ are immediately obvious. For example, we can see from $G \rightarrow \mathbf{C}\, F \vdash_{S5} G \rightarrow \mathbf{C}\,(F \vee H)$ that the logic of causal theories will exhibit the property of 'weakening of the consequent': $F \Leftarrow G$ implies (in a sense to be made more precise) $(F \vee H) \Leftarrow G$. Other properties of $F \Leftarrow G$ will be straightforwardly propositional, such as 'strengthening of the antecedent': $F \Leftarrow G$ implies $F \Leftarrow G \wedge H$. This last property is intriguing, since it is often seen as a characteristic feature of monotonic conditionals, yet the logic of causal theories is nonmonotonic.

In this paper, we will not rely on the translation to Turner's modal logic but prove properties directly from the semantics of causal theories. Although many of the properties can be derived quite straightforwardly in $S5$, there is some additional preliminary notation and terminology that would be needed, and we wish to give a succinct account. Moreover, there are some fundamental properties of causal theories that are not inherited from $S5$.

The second motivation is a practical one. Causal theories and $\mathcal{C}+$ are very expressive languages. One purpose of the technical investigation is to find conditions under which which two different causal theories, or two different action descriptions in $\mathcal{C}+$, can be said to be equivalent, with the further aim of helping to decide between alternative formulations when constructing applications.

2 Causal theories

A *multi-valued propositional signature* σ [GLLT01, GLL$^+$04] is a set of symbols called *constants*. For each constant c in σ, there is a non-empty set $\mathit{dom}(c)$ of values called the *domain* of c. An *atom* of a signature σ is an expression of the form $c{=}v$, where c is a constant in σ and $v \in \mathit{dom}(c)$. A *formula* φ of signature σ is any propositional compound of atoms of σ.

A *Boolean constant* is one whose domain is the set of truth values $\{\mathsf{t}, \mathsf{f}\}$. If p is a Boolean constant, p is shorthand for the atom $p{=}\mathsf{t}$ and $\neg p$ for the atom $p{=}\mathsf{f}$. Notice that, as defined here, $\neg p$ is an *atom* when p is a Boolean constant.

An *interpretation* of σ is a function that maps every constant in σ to an element of its domain. An interpretation I *satisfies* an atom $c{=}v$, written $I \models c{=}v$, if $I(c) = v$. The satisfaction relation $\models$ is extended from atoms to formulas in accordance with the standard truth tables for the propositional connectives. When X is a set of formulas we also write $I \models X$ to signify that $I \models \varphi$ for all formulas $\varphi \in X$. I is then a *model* for the set of formulas X. The set of interpretations of a signature σ will be denoted by $\mathrm{I}(\sigma)$.

We write $\models_\sigma \varphi$ to mean that $I \models_\sigma \varphi$ for all interpretations I of σ. Where X is a set of formulas of signature σ, $X \models_\sigma \varphi$ denotes that $I \models_\sigma \varphi$ for all interpretations I of σ such that $I \models_\sigma X$. When X' is a set of formulas of signature σ, $X \models_\sigma X'$

is shorthand for $X \models_\sigma \varphi$ for all formulas $\varphi \in X'$. In addition, where A and B are sets of formulas of a multi-valued propositional signature, we define $A \equiv_\sigma B$ to mean that $A \models_\sigma B$ and $B \models_\sigma A$. A *causal rule* is an expression of the form $F \Leftarrow G$, where F and G are formulas of signature σ. A *causal theory* is a set of causal rules.

Semantics Let Γ be a causal theory, and let X be an interpretation of its underlying propositional signature. Then the *reduct* of Γ, written Γ^X, is

$$\{F \mid F \Leftarrow G \in \Gamma \text{ and } X \models G\}$$

X is a model of Γ, written $X \models_{\mathsf{C}} \Gamma$, iff X is the unique model of the reduct Γ^X. By $models(\Gamma)$ we denote the set of all models of the causal theory Γ.

Γ^X is the set of all formulas which have a cause to be true, according to the rules of Γ, under the interpretation X. If Γ^X has no models, or has more than one model, or if it has a unique model different from X, then X is not considered to be a model of Γ. Γ is *consistent* or *satisfiable* iff it has a model.

For an illustration of the preceding definitions, consider the causal theory T_1, with underlying Boolean signature $\{p, q\}$:

$$p \Leftarrow q$$
$$q \Leftarrow q$$
$$\neg q \Leftarrow \neg q$$

There are clearly four possible interpretations of the signature:

$$X_1: \; p \mapsto \mathsf{t},\; q \mapsto \mathsf{t}$$
$$X_2: \; p \mapsto \mathsf{t},\; q \mapsto \mathsf{f}$$
$$X_3: \; p \mapsto \mathsf{f},\; q \mapsto \mathsf{t}$$
$$X_4: \; p \mapsto \mathsf{f},\; q \mapsto \mathsf{f}$$

and it is clear that

$$T_1^{X_1} = \{p, q\} \text{ whose only model is } X_1$$
$$T_1^{X_2} = \{\neg q\} \text{ which has two models}$$
$$T_1^{X_3} = \{p, q\} \text{ whose only model is } X_1 \neq X_3$$
$$T_1^{X_4} = \{\neg q\} \text{ which has two models}$$

In only one of these cases—that of X_1—is it true that the reduct of the causal theory with respect to the interpretation has that interpretation as its unique model. Thus $X_1 \models_{\mathsf{C}} T_1$ and $models(T_1) = \{X_1\}$.

Suppose we add another law to T_1: for example, $T_2 = T_1 \cup \{\neg p \Leftarrow \neg p\}$. Now we have:

$$T_2^{X_1} = \{p, q\} \text{ whose only model is } X_1$$
$$T_2^{X_2} = \{\neg q\} \text{ which has two models}$$
$$T_2^{X_3} = \{p, \neg p, q\} \text{ which has no models}$$
$$T_2^{X_4} = \{\neg p, \neg q\} \text{ whose only model is } X_4.$$

Thus, $models(T_2) = \{X_1, X_4\}$; in this example, augmenting the causal theory *increases* the set of models. It is clear that in general, for causal theories Γ and Δ, $models(\Gamma \cup \Delta) \not\subseteq models(\Gamma)$. This is the sense in which the causal theories are nonmonotonic. In the following, one of our purposes will be to invesigate under which conditions $\Gamma \cup \Delta$ has the same models as Γ.

3 A consequence relation between causal theories

In this section, we frequently omit set-theoretic brackets from causal theories where doing so does not create confusion. In particular, causal theories which are singletons are often represented by the sole law they contain.

Proposition 1 $X \models_{\mathsf{C}} \Gamma$ *iff, for every formula* F, $X \models F$ *iff* $\Gamma^X \models_\sigma F$.

Proof: This is Proposition 1 of [GLL+04]. □

Observation 2 $(\Gamma_1 \cup \Gamma_2)^X = \Gamma_1^X \cup \Gamma_2^X$.

Proposition 3 $X \models \{F \Leftarrow G\}^X$ *iff* $X \models G \rightarrow F$.

Proof: Assume $X \models \{F \Leftarrow G\}^X$. If $X \models G$, then $\{F \Leftarrow G\}^X = \{F\}$, so $X \models F$. For the other direction, suppose $X \models G \rightarrow F$. If $X \models G$ then $X \models F$. But then $\{F \Leftarrow G\}^X = \{F\}$ and we have $X \models \{F \Leftarrow G\}^X$. If $X \not\models G$ then $\{F \Leftarrow G\}^X = \emptyset$, and $X \models \{F \Leftarrow G\}^X$, trivially. □

It follows from the above that $X \models \{F_1 \Leftarrow G_1, \ldots, F_n \Leftarrow G_n\}^X$ iff $X \models (G_1 \rightarrow F_1) \wedge \cdots \wedge (G_n \rightarrow F_n)$. Moreover, if a causal theory Γ contains a rule $F \Leftarrow G$ then every model of Γ satisfies $G \rightarrow F$, i.e., $X \models_{\mathsf{C}} \Gamma$ implies $X \models G \rightarrow F$. This last remark is Proposition 2 of [GLL+04].

Where Γ is a causal theory, we will denote by $mat(\Gamma)$ the set of formulas obtained by replacing every rule $F \Leftarrow G$ of Γ by the corresponding material implication, $G \rightarrow F$. The remarks above can thus be summarised as follows.

Proposition 4

$$(i)\ X \models \Gamma^X \text{ iff } X \models mat(\Gamma) \qquad (ii)\ X \models_{\mathsf{C}} \Gamma \text{ implies } X \models mat(\Gamma)$$

Proof: In the preceding discussion. □

We now define a notion of consequence between causal theories. This will allow us to say under which conditions two causal theories are equivalent, to simplify causal theories by removing causal laws that are implied by the causal theory, and to identify (in the following section) general properties of causal laws.

We will say that causal theories Γ_1 and Γ_2 of signature σ are equivalent, written $\Gamma_1 \equiv \Gamma_2$, when $\Delta \cup \Gamma_1$ and $\Delta \cup \Gamma_2$ have the same models for all causal theories Δ of signature σ. We will say that Γ_1 implies Γ_2, written $\Gamma_1 \vdash \Gamma_2$, when $(\Gamma_1 \cup \Gamma_2) \equiv \Gamma_1$, that is, when $\Delta \cup \Gamma_1 \cup \Gamma_2$ and $\Delta \cup \Gamma_1$ have the same models for all causal theories Δ of signature σ.

Proposition 5 $\Gamma_1 \equiv \Gamma_2$ iff $\Gamma_1 \vdash \Gamma_2$ and $\Gamma_2 \vdash \Gamma_1$

Proof: First, suppose $\Gamma_1 \equiv \Gamma_2$, which by definition gives $models(\Gamma \cup \Gamma_1) = models(\Gamma \cup \Gamma_2)$, for all causal theories Γ. Thus clearly $models((\Gamma \cup \Gamma_1) \cup \Gamma_1) = models((\Gamma \cup \Gamma_1) \cup \Gamma_2)$, which is the definition of $\Gamma_1 \vdash \Gamma_2$. We can show $\Gamma_2 \vdash \Gamma_1$ by similar means.
For the other direction, suppose $\Gamma_1 \vdash \Gamma_2$ and $\Gamma_2 \vdash \Gamma_1$. By their definition, these equate both $models(\Gamma \cup \Gamma_1)$ and $models(\Gamma \cup \Gamma_2)$ to $models(\Gamma \cup \Gamma_1 \cup \Gamma_2)$. So they are themselves equal, and this equality defines $\Gamma_1 \equiv \Gamma_2$. □

Proposition 6 $\Gamma \vdash \Gamma_1$ *and* $\Gamma \vdash \Gamma_2$ *iff* $\Gamma \vdash (\Gamma_1 \cup \Gamma_2)$

Proof: Immediate from the definitions. □

Proposition 7 *For all causal theories* Γ, Γ_1, Γ_2, Δ *of signature* σ *we have:*

(*i*) *If* $\Gamma_1 \equiv \Gamma_2$, *then* $(\Gamma_1 \cup \Delta) \vdash \Gamma$ *iff* $(\Gamma_2 \cup \Delta) \vdash \Gamma$.

(*ii*) *If* $\Gamma_1 \equiv \Gamma_2$, *then* $\Gamma \vdash (\Delta \cup \Gamma_1)$ *iff* $\Gamma \vdash (\Delta \cup \Gamma_2)$.

(*iii*) *If* $\Gamma_1 \equiv \Gamma_2$, *then* $(\Gamma_1 \cup \Delta) \equiv (\Gamma_2 \cup \Delta)$.

Proof: Part (*ii*): suppose $\Gamma_1 \equiv \Gamma_2$ and $(\Gamma_1 \cup \Delta) \vdash \Gamma$. That $models(\Delta' \cup (\Gamma_2 \cup \Delta) \cup \Gamma)$ is equal to $models(\Delta' \cup (\Gamma_2 \cup \Delta))$ follows easily using basic set theory. The other parts can be proved in similar fashion. □

Proposition 8 *The relation* $\vdash$ *between causal theories of a given signature* σ *is a classical consequence relation (also known as a closure operator), that is, it satisfies the following three properties, for all causal theories* Γ, Γ_1, *and* Γ_2*:*

- ***inclusion:*** $\Gamma \vdash \Gamma$
- ***cut:*** $(\Gamma_1 \cup \Gamma_2) \vdash \Gamma$ *and* $\Gamma_1 \vdash \Gamma_2$ *implies* $\Gamma_1 \vdash \Gamma$
- ***monotony:*** $\Gamma_1 \vdash \Gamma$ *implies* $(\Gamma_1 \cup \Gamma_2) \vdash \Gamma$

Proof: 'Inclusion' is trivial. For 'cut', first suppose $(\Gamma_1 \cup \Gamma_2) \vdash \Gamma$ and $\Gamma_1 \vdash \Gamma_2$. By definition, these mean that

$$(i)\ (\Gamma_1 \cup \Gamma_2) \cup \Gamma \cup \Delta' \equiv (\Gamma_1 \cup \Gamma_2) \cup \Delta' \qquad (ii)\ \Gamma_1 \cup \Gamma_2 \cup \Delta'' \equiv \Gamma_1 \cup \Delta''$$

So clearly,

$$\begin{aligned}
\Gamma_1 \cup \Gamma \cup \Delta &\equiv \Gamma_1 \cup (\Gamma \cup \Delta) \\
&\equiv \Gamma_1 \cup \Gamma_2 \cup (\Gamma \cup \Delta) && \text{by } (ii) \\
&\equiv \Gamma_1 \cup \Gamma_2 \cup \Delta && \text{by } (i) \\
&\equiv \Gamma_1 \cup \Delta && \text{by } (ii)
\end{aligned}$$

For 'monotony', suppose $\Gamma_1 \vdash \Gamma$. Then $(\Gamma_1 \cup \Gamma) \equiv \Gamma_1$. We show $(\Gamma_1 \cup \Gamma_2 \cup \Gamma) \equiv (\Gamma_1 \cup \Gamma_2)$ for any causal theory Γ_2. Clearly $(\Gamma_1 \cup \Gamma_2 \cup \Gamma) \equiv ((\Gamma_1 \cup \Gamma) \cup \Gamma_2)$. And $((\Gamma_1 \cup \Gamma) \cup \Gamma_2) \equiv (\Gamma_1 \cup \Gamma_2)$ because $(\Gamma_1 \cup \Gamma) \equiv \Gamma_1$. □

Corollary 9 *If* $\Gamma_1 \vdash \Gamma_2$ *and* $\Gamma_2 \vdash \Gamma_3$ *then* $\Gamma_1 \vdash \Gamma_3$.

Proof: If $\Gamma_2 \vdash \Gamma_3$ then by monotony, we have $\Gamma_1 \cup \Gamma_2 \vdash \Gamma_3$. If $\Gamma_1 \vdash \Gamma_2$ and $\Gamma_1 \cup \Gamma_2 \vdash \Gamma_3$ then $\Gamma_1 \vdash \Gamma_3$ by cut. □

Corollary 10 *Let Γ_1, Γ_2, Γ'_1 and Γ'_2 be causal theories of signature σ. Then if $\Gamma_1 \vdash \Gamma_2$ and $\Gamma'_1 \vdash \Gamma'_2$, then $\Gamma_1 \cup \Gamma'_1 \vdash \Gamma_2 \cup \Gamma'_2$.*

Proof: Assume $\Gamma_1 \vdash \Gamma_2$ and $\Gamma'_1 \vdash \Gamma'_2$. From the latter, by monotony, we have $(\Gamma_1 \cup \Gamma'_1) \cup \Gamma_2 \vdash \Gamma'_2$; from the former, also by monotony, we have $(\Gamma_1 \cup \Gamma'_1) \vdash \Gamma_2$. Now, by inclusion we have that $\Gamma_2 \cup \Gamma'_2 \vdash \Gamma_2 \cup \Gamma'_2$, and so by monotony, $(\Gamma_1 \cup \Gamma'_1) \cup \Gamma_2 \cup \Gamma'_2 \vdash \Gamma_2 \cup \Gamma'_2$. We now apply cut twice, using the results established by monotony at the beginning.

$$\frac{(\Gamma_1 \cup \Gamma'_1) \cup \Gamma_2 \cup \Gamma'_2 \vdash \Gamma_2 \cup \Gamma'_2, \qquad (\Gamma_1 \cup \Gamma'_1) \cup \Gamma_2 \vdash \Gamma'_2}{(\Gamma_1 \cup \Gamma'_1) \cup \Gamma_2 \vdash \Gamma_2 \cup \Gamma'_2};$$

and for the second application of cut,

$$\frac{(\Gamma_1 \cup \Gamma'_1) \cup \Gamma_2 \vdash \Gamma_2 \cup \Gamma'_2, \qquad (\Gamma_1 \cup \Gamma'_1) \vdash \Gamma_2}{\Gamma_1 \cup \Gamma'_1 \vdash \Gamma_2 \cup \Gamma'_2}$$

□

Notice that although the formalism of causal theories is nonmonotonic, in the sense that in general $\mathit{models}(\Gamma \cup \Delta) \not\subseteq \mathit{models}(\Gamma)$, the consequence relation $\vdash$ between causal theories is monotonic.

We now establish some simple *sufficient* conditions under which $\vdash$ holds.

Proposition 11 *$\mathit{models}(\Gamma_1) \subseteq \mathit{models}(\Gamma_2)$ iff, for all $X \in \mathit{models}(\Gamma_1)$, we have $\Gamma_1^X \equiv_\sigma \Gamma_2^X$.*

Proof: Suppose $\mathit{models}(\Gamma_1) \subseteq \mathit{models}(\Gamma_2)$ and $X \in \mathit{models}(\Gamma_1)$. Then $X \in \mathit{models}(\Gamma_2)$ also. Now consider any interpretation Y. $X \models_{\mathsf{C}} \Gamma_1$, so $Y \models \Gamma_1^X$ iff $Y = X$. $X \models_{\mathsf{C}} \Gamma_2$ so $Y \models \Gamma_2^X$ iff $Y = X$. It follows that $Y \models \Gamma_1^X$ iff $Y \models \Gamma_2^X$, as required.
For the other half, suppose $\Gamma_1^X \equiv_\sigma \Gamma_2^X$ for all $X \in \mathit{models}(\Gamma_1)$. Further suppose $Y \in \mathit{models}(\Gamma_1)$. Let Z be any interpretation. Since Y is a model of Γ_1, $Z \models \Gamma_1^Y$ iff $Z = Y$. But $Y \in \mathit{models}(\Gamma_1)$, so $\Gamma_1^Y \equiv_\sigma \Gamma_2^Y$, and hence $Z \models \Gamma_1^Y$ iff $Z \models \Gamma_2^Y$. So we have $Z \models \Gamma_2^Y$ iff $Z = Y$, i.e. $Y \in \mathit{models}(\Gamma_2)$, as required. □

Corollary 12 *$\mathit{models}(\Gamma_1) = \mathit{models}(\Gamma_2)$ iff we have $\Gamma_1^X \equiv_\sigma \Gamma_2^X$, for all $X \in \mathit{models}(\Gamma_1) \cup \mathit{models}(\Gamma_2)$.*

Proposition 13

(*i*) *$\Gamma_1 \vdash \Gamma_2$ if $\Gamma_1^X \models_\sigma \Gamma_2^X$, for all $X \models \mathit{mat}(\Gamma_1 \cup \Gamma_2)$.*

(*ii*) *$\Gamma_1 \vdash \Gamma_2$ if $\Gamma_1^X \models_\sigma \Gamma_2^X$, for all $X \models \mathit{mat}(\Gamma_1)$.*

(*iii*) *$\Gamma \vdash (F \Leftarrow G)$ if $\Gamma^X \models_\sigma F$, for all $X \models \mathit{mat}(\Gamma_1) \cup \{G\}$.*

Proof: Part (*i*) follows from considering Proposition 11 and Corollary 12; the details of this have been omitted. Part (*ii*) is obtained from Part (*i*) by strengthening the condition. Part (*iii*) follows from Part (*ii*): if $X \models G$ then $\{F \Leftarrow G\}^X = \{F\}$. If $X \not\models G$ then $\{F \Leftarrow G\}^X = \emptyset$, and so $\Gamma^X \models_\sigma \{F \Leftarrow G\}^X$, trivially. □

We record one further property for future reference. A causal rule of the form $F \Leftarrow F$ expresses that F holds by default. Adding $F \Leftarrow F$ to a causal theory Γ cannot eliminate models, though it can add to them.

Proposition 14 $models(\Gamma) \subseteq models(\Gamma \cup \{F \Leftarrow F\})$

Proof: Suppose $X \models_{\mathsf{C}} \Gamma$, i.e., $X \models \Gamma^X$ and $Y \models \Gamma^X$ implies $Y = X$. We show (i) $X \models (\Gamma \cup \{F \Leftarrow F\})^X$, and (ii) if $Y \models (\Gamma \cup \{F \Leftarrow F\})^X$ then $Y = X$. For (i): if $X \models F$ then $(\Gamma \cup \{F \Leftarrow F\})^X = \Gamma^X \cup \{F\}$; we have both $X \models \Gamma^X$ and $X \models F$. If $X \not\models F$, then $(\Gamma \cup \{F \Leftarrow F\})^X = \Gamma^X$; we have $X \models \Gamma^X$. For (ii), suppose $Y \models (\Gamma \cup \{F \Leftarrow F\})^X$. Then $Y \models \Gamma^X$, and $X \models_{\mathsf{C}} \Gamma$ implies $Y = X$. □

4 Properties of $\Leftarrow$

We can now prove properties about the logic of causal theories, using the preliminary results and definitions given in the previous section. We have chosen to name the results after Chellas's [Che80] taxonomy of rules of inference from modal logic, as this scheme is well-known and seems natural to us. The reader may care to check that the propositions and corollaries presented here are all reasonable given a reading of $F \Leftarrow G$ as 'if G, then there is a cause for F', though *not* reasonable for the stronger interpretation, 'G causes F'.

In the following, we will frequently use the notational convenience of writing $\frac{A}{B}$ instead of $A \vdash B$, where A and B are causal rules or sets of such.

Proposition 15 [RCM] If $F_1 \models_\sigma F_2$, then $F_1 \Leftarrow G \ \vdash\ F_2 \Leftarrow G$

Proof: From Proposition 13(*iii*), a sufficient condition for $F_1 \Leftarrow G \vdash F_2 \Leftarrow G$ is $\{F_1 \Leftarrow G\}^X \models_\sigma F_2$ for all $X \models_\sigma G$, which is just $F_1 \models_\sigma F_2$, which was given. □

Proposition 16 [RAug] If $G_1 \models_\sigma G_2$, then $F \Leftarrow G_2 \ \vdash\ F \Leftarrow G_1$

Proof: Similar to that for Proposition 15, and also using Proposition 13(*ii*). □

Given the preceding two propositions and Proposition 5, we have the following corollary. Naming conventions again follow [Che80].

Corollary 17

[RCEC]	(*i*) If $F_1 \equiv_\sigma F_2$, then $F_1 \Leftarrow G \equiv F_2 \Leftarrow G$
[RCEA]	(*ii*) If $G_1 \equiv_\sigma G_2$, then $F \Leftarrow G_1 \equiv F \Leftarrow G_2$

Proposition 18

$$\text{[RCK]} \qquad \text{If } F_1, \ldots, F_n \models_\sigma F, \text{ then } \quad \frac{F_1 \Leftarrow G, \ldots, F_n \Leftarrow G}{F \Leftarrow G} \ (n \geqslant 0)$$

Proof: For the case $n = 0$, a sufficient condition for $\vdash F \Leftarrow G$ is $\emptyset^X \models_\sigma F$ for all X, which holds, since $\models_\sigma F$ was given.
For the general case, a sufficient condition for $F_1 \Leftarrow G, \ldots, F_n \Leftarrow G \vdash F \Leftarrow G$ is $\{F_1 \Leftarrow G, \ldots, F_n \Leftarrow G\}^X \models_\sigma F$ for all X such that $X \models G$, which is $F_1, \ldots, F_n \models_\sigma F$, which was given. □

The above are properties characteristic of 'normal conditional logics' [Che80]. We now move on to consider some distribution laws.

Proposition 19

$$[\text{CC}] \quad \frac{F_1 \Leftarrow G, \ldots, F_n \Leftarrow G}{(F_1 \wedge \cdots \wedge F_n) \Leftarrow G} \qquad [\text{CM}] \quad \frac{(F_1 \wedge \cdots \wedge F_n) \Leftarrow G}{F_1 \Leftarrow G, \ldots, F_n \Leftarrow G}$$

$$[\text{DIL}] \quad \frac{F \Leftarrow G_1, \ldots, F \Leftarrow G_n}{F \Leftarrow (G_1 \vee \cdots \vee G_n)} \qquad [\text{cDIL}] \quad \frac{F \Leftarrow (G_1 \vee \cdots \vee G_n)}{F \Leftarrow G_1, \ldots, F \Leftarrow G_n}$$

Proof: [CC]. We have $\{F_1 \Leftarrow G, \ldots, F_n \Leftarrow G\}^X \models_\sigma (F_1 \wedge \cdots \wedge F_n)$, for all $X \models G$, as a sufficient condition. This clearly holds.
[CM]. From an easy generalization of Proposition 13*(ii)* a sufficient condition is that $\{(F_1 \wedge \cdots \wedge F_n) \Leftarrow G\}^X \models_\sigma \{F_1 \Leftarrow G, \ldots, F_n \Leftarrow G\}^X$ for all interpretations X. If $X \models G$, our condition reduces to $(F_1 \wedge \cdots \wedge F_n) \models_\sigma \{F_1, \ldots, F_n\}$, which we clearly have. Otherwise, if $X \not\models G$, then it reduces to $\emptyset \models_\sigma \emptyset$.
[DIL]. A sufficient condition is $\{F \Leftarrow G_1, \ldots, F \Leftarrow G_n\}^X \models_\sigma F$ for all $X \models (G_1 \vee \cdots \vee G_n)$. Yet if $X \models (G_1 \vee \cdots \vee G_n)$ then $X \models G_i$ for some $1 \leqslant i \leqslant n$. Hence $\{F \Leftarrow G_1, \ldots F \Leftarrow G_n\}^X = \{F\}$, so we require only $F \models_\sigma F$, which holds.
[cDIL]. A sufficient condition for $F \Leftarrow (G_1 \vee \cdots \vee G_n) \vdash F \Leftarrow G_i$, for every $1 \leqslant i \leqslant n$, is $\{F \Leftarrow (G_1 \vee \cdots \vee G_n)\}^X \models_\sigma F$ for all $X \models G_i$. But if $X \models G_i$, then $\{F \Leftarrow (G_1 \vee \cdots \vee G_n)\}^X = \{F\}$, and again we merely require $F \models_\sigma F$. □

There follows a network of interrelated properties which all express a form of monotonicity of the conditional $\Leftarrow$.

Proposition 20 [Aug] $\quad F \Leftarrow G \ \vdash \ F \Leftarrow G \wedge H$

Proof: This is clearly a specific instance of [RAug]. For a direct proof: a sufficient condition for [Aug] is $\{F \Leftarrow G\}^X \models_\sigma F$ for all $X \models (G \wedge H)$. But if $X \models (G \wedge H)$, then the condition reduces to $F \models_\sigma F$, which holds. □

In fact, it can be shown that in the presence of the rule [RCEA], which we proved as Corollary 17*(ii)*, the schema [Aug] is equivalent to the distribution law [cDIL]:

Proposition 21 *If a conditional logic contains the rule* [RCEA]*, then it contains the schema* [Aug] *iff it contains the schema* [cDIL]*.*

Proof: First, a derivation of Aug from RCEA and cDIL:

1.	$F \Leftarrow G$	ass.
2.	$F \Leftarrow G \wedge (H \vee \neg H)$	(1, PL, RCEA)
3.	$F \Leftarrow (G \wedge H) \vee (G \wedge \neg H)$	(2, PL, RCEA)
4.	$F \Leftarrow G \wedge H$	(3, cDIL)

For the other direction, a derivation of cDIL from RCEA and Aug (it is sufficient, without loss of generality, to deal with the case $n = 2$):

1.	$F \Leftarrow G_1 \vee G_2$	ass.
2.	$F \Leftarrow (G_1 \vee G_2) \wedge (G_1 \vee \neg G_2)$	(1, Aug)
3.	$(G_1 \vee G_2) \wedge (G_1 \vee \neg G_2) \equiv_{PL} G_1$	(PL)
4.	$F \Leftarrow G_1$	(2, 3, RCEA)

□

Proposition 22 [Contra] $\vdash F \Leftarrow \bot$

Proof: A sufficient condition for this is $\emptyset^X \vdash F$ for all $X \models \bot$, which holds trivially, since there is no such X. □

Proposition 23 $F \Leftarrow G \ \vdash \ \bot \Leftarrow \neg F \wedge G$

Proof: By Proposition 13*(iii)*, a sufficient condition is that $\{F \Leftarrow G\}^X \models_\sigma \bot$, for all X with $X \models (\neg F \wedge G) \wedge (G \to F)$, which obtains: there is no such X. □

The converse of this proposition does not hold: $\bot \Leftarrow \neg F \wedge G \not\vdash F \Leftarrow G$.

Now, since $G \models_\sigma \bot$ iff $G \equiv_\sigma \bot$, the schema [Contra] is equivalent to the rule: if $G \models_\sigma \bot$ then $\vdash F \Leftarrow G$. This is the case $n = 0$ for the following generalization of [RAug].

Proposition 24

$$\text{[RDIL]} \qquad \text{If } G \models_\sigma (G_1 \vee \cdots \vee G_n), \text{ then } \quad \frac{F \Leftarrow G_1, \ldots, F \Leftarrow G_n}{F \Leftarrow G} \ (n \geqslant 0)$$

Proof: The case for $n = 0$ is covered by the schema [Contra].
For $n > 0$, first suppose $G \models_\sigma (G_1 \vee \cdots \vee G_n)$. A sufficient condition for $F \Leftarrow G_1, \ldots, F \Leftarrow G_n \vdash F \Leftarrow G$ is $\{F \Leftarrow G_1, \ldots, F \Leftarrow G_n\}^X \models_\sigma F$ for all $X \models G$. But $X \models G$ implies $X \models G_i$ for some $1 \leqslant i \leqslant n$, by our hypothesis. Thus $\{F \Leftarrow G_1, \ldots, F \Leftarrow G_n\}^X = \{F\}$, and we now only require $F \models_\sigma F$. □

We can show an equivalence between RDIL and rules already proven.

Proposition 25 [DIL] *and* [Aug] *(equivalently,* [DIL] *and* [cDIL]*) are equivalent to the rule* [RDIL] *for* $n \geqslant 1$.

Proof: [RAug] is the special case of [RDIL] where $n = 1$. The distribution law [DIL] follows from [RDIL] and $(G_1 \vee \cdots \vee G_n) \models_\sigma (G_1 \vee \cdots \vee G_n)$.
For the other direction, [RDIL] ($n \geqslant 1$) can be derived from [Aug] and [DIL] as follows:

1.	$G \models_\sigma (G_1 \vee \cdots \vee G_n)$	ass.
2.	$F \Leftarrow G_1, \ldots, F \Leftarrow G_n$	ass.
3.	$F \Leftarrow (G_1 \wedge G), \ldots, F \Leftarrow (G_n \wedge G)$	(2, Aug)
4.	$F \Leftarrow (G_1 \wedge G) \vee \cdots \vee (G_n \wedge G)$	(3, DIL)
5.	$(G_1 \wedge G) \vee \cdots \vee (G_n \wedge G) \equiv_\sigma G$	(1, PL)
6.	$F \Leftarrow G$	(3, 5, RCEA)

□

Proposition 26 [S] $F \Leftarrow G, \ G \Leftarrow H \ \vdash \ F \Leftarrow H$

Proof: By Proposition 13(*iii*), we have that a sufficient condition for this is $\{F \Leftarrow G, G \Leftarrow H\}^X \models_\sigma F$, for all X such that $X \models H \wedge (G \to F) \wedge (H \to G)$. But every such X has $X \models G \wedge H$, so that $\{F \Leftarrow G, G \Leftarrow H\}^X = \{F, G\}$, and we require only $\{F, G\} \models_\sigma F$. □

A statement of the propogation of constraints, and a rule of Modus Ponens, are obvious instances of [S]:

Corollary 27

$$\text{[Constr]} \qquad \frac{F \Leftarrow G, \quad \bot \Leftarrow F}{\bot \Leftarrow G}, \qquad\qquad \text{[MP]} \qquad \frac{F \Leftarrow G, \quad G \Leftarrow \top}{F \Leftarrow \top}$$

The following is a generalization of [Contra].

Proposition 28 $G \Leftarrow \top \ \vdash \ F \Leftarrow \neg G$

Proof: From $G \Leftarrow \top$, we get $\bot \Leftarrow \neg G \wedge \top$ using Proposition 23; an application of [RCEA] then gives us $\bot \Leftarrow \neg G$. Using [Contra] and [S] we then derive $F \Leftarrow \neg G$: □

The rule describing propogation of constraints may be generalised to a weak resolution law for Horn-like rules:

Proposition 29 $F \Leftarrow G \wedge H, \ G \Leftarrow K \ \vdash \ F \Leftarrow H \wedge K$

Proof: We have $G \Leftarrow K \vdash \bot \Leftarrow K \wedge \neg G$, from Proposition 23, and so using [Contra] and [S] we have also $F \Leftarrow K \wedge \neg G$; from the latter using [Aug] we get $F \Leftarrow \neg G \wedge H \wedge K$. From $F \Leftarrow G \wedge H$, also using [Aug], we have $F \Leftarrow G \wedge H \wedge K$. Applying [Dil], we get

$$F \Leftarrow (\neg G \wedge H \wedge K) \vee (G \wedge H \wedge K),$$

which using [RCEA] gives us the desired $F \Leftarrow K \wedge H$. □

We mention resolution here as the following is clearly a special case:

Corollary 30

$$\frac{\bot \Leftarrow G \wedge H, \quad G \Leftarrow K}{\bot \Leftarrow K \wedge H}$$

The logic of causal theories does *not* contain the two equivalent rules

$$\text{[I]} \qquad \vdash F \Leftarrow F; \qquad\qquad \text{[RI]} \qquad \text{If} \quad G \models_\sigma F, \quad \text{then} \quad \vdash F \Leftarrow G$$

To see this, use $\neg p \Leftarrow \neg p$ for F and consider the causal theory with the single rule $p \Leftarrow p$; $models(\{p \Leftarrow p\}) \neq models(\{p \Leftarrow p, \neg p \Leftarrow \neg p\})$ which means that we do not have $models(\Gamma \cup \emptyset) = models(\Gamma \cup \emptyset \cup \{\neg p \Leftarrow \neg p\})$ for all causal theories Γ, and so $\nvdash \neg p \Leftarrow \neg p$. Although we do not have [I], it has already been shown (Proposition 14) that $models(\Gamma) \subseteq models(\Gamma \cup \{F \Leftarrow F\})$.

Using the same example, it can easily be seen that the logic of $\Leftarrow$ does not contain a contrapositive law: we have that, in general, $F \Leftarrow G \nvdash \neg G \Leftarrow \neg F$. This is exactly what we would expect given the intended informal reading of $F \Leftarrow G$.

Example As one example of an application of these properties consider the following common patterns of causal rules:

$$\{\, F \Leftarrow F \wedge G \wedge \neg R, \ \neg F \Leftarrow R \,\} \qquad \text{and} \qquad \{\, F \Leftarrow F \wedge G, \ \neg F \Leftarrow R \,\}$$

In each case the first law expresses that F holds by default if G holds, and the second that R is an exception to the default rule. These pairs of laws are equivalent in causal theories. One direction is straightforward: $F \Leftarrow F \wedge G \wedge \neg R$ follows from $F \Leftarrow F \wedge G$ by [Aug].

For the other direction, notice first that $\neg F \Leftarrow R$ implies $\bot \Leftarrow F \wedge G \wedge R$ (because $\neg F \Leftarrow R$ implies $\bot \Leftarrow F \wedge R$ and the rest follows by [Aug]). Now $\vdash F \Leftarrow \bot$ [Contra], and by [S] we derive $F \Leftarrow F \wedge G \wedge R$. For the final step

$$\frac{F \Leftarrow F \wedge G \wedge R, \quad F \Leftarrow F \wedge G \wedge \neg R}{F \Leftarrow F \wedge G \wedge (R \vee \neg R)}$$

from which $F \Leftarrow F \wedge G$ follows.

5 The action language $\mathcal{C}+$

5.1 Syntax

As with the logic of causal theories, the language $\mathcal{C}+$ is based on a multi-valued propositional signature σ, with σ partitioned into a set σ^f of *fluent constants* and a set σ^a of *action constants*. Further, the fluent constants are partitioned into those which are *simple* and those which are *statically determined*. A *fluent formula* is a formula whose constants all belong to σ^f; an action formula has at least one action constant and no fluent constants.

A *static law* is an expression of the form

$$\textsf{caused } F \textsf{ if } G,$$

where F and G are fluent formulas. An *action dynamic law* is an expression of the same form in which F is an action formula and G is a formula. A *fluent dynamic law* has the form

$$\textsf{caused } F \textsf{ if } G \textsf{ after } H,$$

where F and G are fluent formulas and H is a formula, with the restriction that F must not contain statically determined fluents. *Causal laws* are static laws or dynamic laws, and an *action description* is a set of causal laws.

In the following section we will make use of several of the many abbreviations afforded in $\mathcal{C}+$. In particular:

α causes F if G abbreviates the fluent dynamic law caused F if $\top$ after $\alpha \wedge G$;

nonexecutable α if G expresses that there is no transition of type α from a state satisfying fluent formula G. It is shorthand for the fluent dynamic law caused $\bot$ if $\top$ after $\alpha \wedge G$;

inertial f where f is a simple fluent constant, states that the values of f persist by default—they are subject to inertia—from one state to the next. It stands for the collection of fluent dynamic laws caused $f{=}v$ if $f{=}v$ after $f{=}v$ for every $v \in dom(f)$.

exogenous a where a is an action constant, stands for the set of action dynamic laws caused $a{=}v$ if $a{=}v$ for every $v \in dom(a)$.

5.2 Semantics

The language $\mathcal{C}+$ can be viewed as a useful shorthand for the logic of causal theories, for to every action description D of $\mathcal{C}+$ and non-negative integer m, there corresponds a causal theory Γ^D_m.

The signature of Γ^D_m contains constants $c[i]$, such that

- $i \in \{0, \ldots, m\}$ and c is a fluent constant of the signature of D, or
- $i \in \{0, \ldots, m{-}1\}$ and c is an action constant of the signature of D,

and the domains of such constants $c[i]$ are kept identical to those of their constituents c. Where σ is the signature of D, we let σ_m denote the signature of Γ^D_m. The expression $F[i]$, where F is a formula, denotes the result of suffixing $[i]$ to every occurrence of a constant in F.

Proposition 31 *Let D be an action description of $\mathcal{C}+$ with signature σ. Let m and i be non-negative integers such that $0 \leqslant i \leqslant m$. For any formulas F and G such that $F[i]$ and $G[i]$ are well defined, we have $F \models_\sigma G$ iff $F[i] \models_{\sigma_m} G[i]$.*

Proof: (The 'if' part.) Assume $F[i] \models_{\sigma_m} G[i]$, so that for all interpretations $I^* \in \mathrm{I}(\sigma_m)$, we have that if $I^* \models_{\sigma_m} F[i]$, then $I^* \models_{\sigma_m} G[i]$. We must show $F \models_\sigma G$, i.e. that for $I \in \mathrm{I}(\sigma)$, if $I \models_\sigma F$ then $I \models_\sigma G$. Assume for some I that $I \models_\sigma F$. We choose an interpretation $I^* \in \mathrm{I}(\sigma_m)$ such that $I^*(c[i]) = v$ iff $I(c) = v$, and where $I^*(c[j])$ (where $j \neq i$) maps to any $v \in dom(c[j])$. Clearly, by structural induction, we have that for all formulas H of signature σ, $I \models_\sigma H$ iff $I^* \models_{\sigma_m} H[i]$. Thus $I^* \models_{\sigma_m} F[i]$, and so by assumption, $I^* \models_{\sigma_m} G[i]$. Then clearly $I \models_\sigma G$, and the 'if' part is done.
The 'only if' part is similar. □

The causal rules of Γ^D_m are:

$$F[i] \Leftarrow G[i],$$

for every static law in D and every $i \in \{0, \ldots, m\}$, and for every action dynamic law in D and every $i \in \{0, \ldots, m{-}1\}$;

$$F[i+1] \Leftarrow G[i+1] \wedge H[i],$$

for every fluent dynamic law in D and every $i \in \{0, \ldots, m{-}1\}$; and

$$f[0]{=}v \Leftarrow f[0]{=}v,$$

for every simple fluent constant f and $v \in dom(c)$.

Each action description of $\mathcal{C}+$ defines a labelled transition system; the definition uses the translation of action descriptions into causal theories described above. Let us suppose we have an action description D, with signature composed of $\sigma^f \cup \sigma^a$. Interpretations of the underlying propositional signature of D are identified with the sets of atoms they satisfy. Thus, where i is a non-negative integer and s an interpretation, we can write $s[i]$ for the result of suffixing $[i]$ to the constant in every atom satisfied by the interpretation (in symbols, $s[i] = \{c[i] = v \mid s \models c{=}v\}$.

The vertices of the transition system defined by D are states: interpretations s of σ^f, such that $s[0]$ is a model of Γ^D_0. The edges of the transition system are triples (s, e, s'), where s and s' are interpretations of σ^f and e is an interpretation of σ^a, and such that $s[0] \cup e[0] \cup s'[1]$ is a model of Γ^D_1. These triples are known as *transitions*, and the component e will sometimes be called a *transition label* or an *event*. Further, when α is a formula of signature σ^a, we say that a transition label/event e is of type α when $e \models \alpha$.

Let Γ^D_m be the causal theory generated from the action description D and non-negative integer m as described above. Let $s_0, \ldots, s_m$ be interpretations of

σ^f and $e_0, \ldots, e_{m-1}$ be interpretations of σ^a. Then using the notation above, interpretations of the signature of Γ_m^D can be represented in the form

$$s_0[0] \cup e_0[0] \cup s_1[1] \cup e_1[1] \cup \cdots \cup e_{m-1}[m-1] \cup s_m[m]. \qquad (1)$$

Proposition 32 *An interpretation of the signature of Γ_m^D is a model of Γ_m^D iff each triple (s_i, e_i, s_{i+1}), for $0 \leqslant i < m$, is a transition.*

Proof: Proposition 8 of [GLL+04]. □

Let D be an action description of $\mathcal{C}+$. A *run* of length m through this transition system is defined to be a sequence

$$(s_0, e_0, s_1, e_1, \ldots, e_{m-1}, s_m) \qquad (2)$$

such that all triples (s_i, e_i, s_{i+1}), for $0 \leqslant i < m$, are members of the transition system.

Proposition 33 *Let D be an action description and m any non-negative integer. Then the sequence* (2) *is a run of the transition system iff the interpretation* (1) *is a model of the causal theory Γ_m^D.*

Proof: First, assume we have a run of the transition system of length m. Then every triple (s_i, e_i, s_{i+1}), for $0 \leqslant i < m$, is a transition, and so by Proposition 32 the interpretation (1) is a model of Γ_m^D.
Alternately, suppose that (1) is a model of the causal theory Γ_m^D. Then clearly, each triple (s_i, e_i, s_{i+1}), for $0 \leqslant i < m$, is a transition, and so the sequence (2) is a run of the transition system defined by D. □

5.3 A consequence relation between $\mathcal{C}+$ action descriptions

As we did for the logic of causal theories, we now define a consequence relation between action descriptions of $\mathcal{C}+$ having the same signature. Further, since action descriptions of $\mathcal{C}+$ may be viewed as shorthand for families of causal theories, the definition of this conequence relation supervenes directly on that defined earlier (Section 3) for causal theories. Again, our objective is to see under which conditions causal laws are redundant, when one theory includes another, and when given laws are implied (according to our consequence relation) by an action description.

Let D_1 and D_2 be two action descriptions of $\mathcal{C}+$, with the same signature σ. Then, $D_1 \vdash_{\mathcal{C}+} D_2$ is defined to hold when $\Gamma_m^{D_1} \vdash \Gamma_m^{D_2}$ for all non-negative integers m. Similarly, we define that $D_1 \equiv_{\mathcal{C}+} D_2$ shall mean $\Gamma_m^{D_1} \equiv \Gamma_m^{D_2}$.

Proposition 34 *Let D_1, D_2 and D be action descriptions of $\mathcal{C}+$ with the same signature σ. Then*

(*i*) *If $D_1 \vdash_{\mathcal{C}+} D_2$, then the labelled transition system defined by $D \cup D_1$ is the same as that defined by $D \cup D_1 \cup D_2$.*

(*ii*) *If $D_1 \equiv_{\mathcal{C}+} D_2$, then the transition system defined by $D \cup D_1$ is the same as that defined by $D \cup D_2$.*

Proof: For part (*i*): assume $D_1 \vdash_{\mathcal{C}+} D_2$. Then by definition $\Gamma_m^{D_1} \vdash \Gamma_m^{D_2}$, for all non-negative integers m. This implies, by further definitions, that

$$models(\Gamma_m^{D \cup D_1 \cup D_2}) = models(\Gamma_m^{D \cup D_1})$$

for all non-negative m and action descriptions D with signature σ. Substituting 0 for m here immediately gives us that the states of the transition systems defined by $D \cup D_1$ are the same to those defined by $D \cup D_1 \cup D_2$, by definition. That the edges of the labelled transition systems are identical easily follows by substituting 1 for m and again considering the definitions.
For (ii): assume $D_1 \equiv_{\mathcal{C}+} D_2$. Then $\Gamma_m^{D_1} \equiv \Gamma_m^{D_2}$ for all non-negative m, by definition. By Proposition 5 we have that $\Gamma_m^{D_1} \vdash \Gamma_m^{D_2}$ and $\Gamma_m^{D_2} \vdash \Gamma_m^{D_1}$, for all $m \geqslant 0$. Using part (i), this means that the transition systems defined by $D \cup D_1$, $D \cup D_1 \cup D_2$ and $D \cup D_2$ are all equal, which gives us the desired result. □

6 Logical properties of $\mathcal{C}+$

As was the case for the logic of causal theories, we will in this section often write $\frac{A}{B}$ instead of $A \vdash_{\mathcal{C}+} B$. We will also often omit the brackets around action descriptions, and sometimes omit the word caused from the beginning of causal laws for typographic convenience.

The proofs for theorems in this section follow the same pattern, which involves moving from the action descriptions of $\mathcal{C}+$ to the underlying representation of causal theories, then using one of the rules of inference established in Section 4, and finally moving back to action descriptions and $\mathcal{C}+$. We give the first proof in detail to show how this strategy works, but details for the later propositions are omitted.

Proposition 35 *If $F_1 \models_\sigma F_2$, then*

$$(i) \quad \frac{\textsf{caused } F_1 \textsf{ if } G}{\textsf{caused } F_2 \textsf{ if } G} \qquad\qquad (ii) \quad \frac{\textsf{caused } F_1 \textsf{ if } G \textsf{ after } H}{\textsf{caused } F_2 \textsf{ if } G \textsf{ after } H}$$

Proof: (Part (i).) Assume $F_1 \models_\sigma F_2$. Thus by Proposition 31 we have that for all non-negative integers m and i such that $i \leqslant m$, if $F_1[i]$ and $F_2[i]$ are defined, then $F_1[i] \models_{\sigma_m} F_2[i]$. Thus where G is a formula of σ such that $G[i]$ is a formula of signature σ_m, we have using [RCM] that $F_1[i] \Leftarrow G[i] \vdash F_2[i] \Leftarrow G[i]$, for all i as assumed. Using Corollary 10 we then have that $\{F_1[i] \Leftarrow G[i] \mid 0 \leqslant i \leqslant m\} \vdash \{F_2[i] \Leftarrow G[i] \mid 0 \leqslant i \leqslant m\}$, which gives us $\Gamma_m^{\{F_1 \textsf{ if } G\}} \vdash \Gamma_m^{\{F_2 \textsf{ if } G\}}$, for all non-negative m.
(Part (ii).) The proof for this part is similar, also using [RCM]. □

Proposition 36 *If $G_1 \models_\sigma G_2$ and $H_1 \models_\sigma H_2$, then*

$$(i) \quad \frac{\textsf{caused } F \textsf{ if } G_2}{\textsf{caused } F \textsf{ if } G_1} \qquad\qquad (ii) \quad \frac{\textsf{caused } F \textsf{ if } G_2 \textsf{ after } H}{\textsf{caused } F \textsf{ if } G_1 \textsf{ after } H}$$

$$(iii) \quad \frac{\textsf{caused } F \textsf{ if } G \textsf{ after } H_2}{\textsf{caused } F \textsf{ if } G \textsf{ after } H_1}$$

Proof: (Part (i).) Use [RAug], Proposition 31 and Corollary 10, according to the pattern illustrated in Proposition 35.
(Part (ii).) Assume $G_1 \models_\sigma G_2$. Then using Proposition 31, for all non-negative integers m and all i such that $0 \leqslant i < m$, we have $G_1[i+1] \models_{\sigma_m} G_2[i+1]$, and so too $G_1[i+1] \wedge H[i] \models_{\sigma_m} G_2[i+1] \wedge H[i]$. Thus using [RAug], we have for m

and i as constrained, $F[i+1] \Leftarrow G_1[i+1] \wedge H[i] \models_{\sigma_m} F[i+1] \Leftarrow G_2[i+1] \wedge H[i]$. Using Corollary 10 as before, we have our result.
(Part (*iii*).) The proof for this part is similar. □

Proposition 37 *If* $F_1 \equiv_\sigma F_2$*, then*

$$(i) \quad \mathsf{caused}\ F_1\ \mathsf{if}\ G \equiv_{\mathcal{C}+} \mathsf{caused}\ F_2\ \mathsf{if}\ G$$
$$(ii) \quad \mathsf{caused}\ F_1\ \mathsf{if}\ G\ \mathsf{after}\ H \equiv_{\mathcal{C}+} \mathsf{caused}\ F_2\ \mathsf{if}\ G\ \mathsf{after}\ H$$

Proof: From [RCEC], according to the pattern established. □

Proposition 38 *If* $G_1 \equiv_\sigma G_2$ *and* $H_1 \equiv_\sigma H_2$*, then*

$$(i) \quad \mathsf{caused}\ F\ \mathsf{if}\ G_1 \equiv_{\mathcal{C}+} \mathsf{caused}\ F\ \mathsf{if}\ G_2$$
$$(ii) \quad \mathsf{caused}\ F\ \mathsf{if}\ G_1\ \mathsf{after}\ H_1 \equiv_{\mathcal{C}+} \mathsf{caused}\ F\ \mathsf{if}\ G_2\ \mathsf{after}\ H_2$$

Proof: From [RCEA], Proposition 31, Corollary 10, and the definition of the translation into causal theories as before. □

Proposition 39 *If* $F_1, \ldots, F_n \models_\sigma F$*, then*

$$(i) \quad \frac{\mathsf{caused}\ F_1\ \mathsf{if}\ G, \ldots, \mathsf{caused}\ F_n\ \mathsf{if}\ G}{\mathsf{caused}\ F\ \mathsf{if}\ G}$$

$$(ii) \quad \frac{\mathsf{caused}\ F_1\ \mathsf{if}\ G\ \mathsf{after}\ H, \ldots, \mathsf{caused}\ F_n\ \mathsf{if}\ G\ \mathsf{after}\ H}{\mathsf{caused}\ F\ \mathsf{if}\ G\ \mathsf{after}\ H}$$

Proof: Using [RCK]. □

Proposition 40

$$(i)\ F_1\ \mathsf{if}\ G, \ldots, F_n\ \mathsf{if}\ G \equiv_{\mathcal{C}+} F_1 \wedge \cdots \wedge F_n\ \mathsf{if}\ G$$
$$(ii)\ F_1\ \mathsf{if}\ G\ \mathsf{after}\ H, \ldots, F_n\ \mathsf{if}\ G\ \mathsf{after}\ H \equiv_{\mathcal{C}+} F_1 \wedge \cdots \wedge F_n\ \mathsf{if}\ G\ \mathsf{after}\ H$$

Proof: From [CC] and [CM]. □

Proposition 41

$$(i)\ F\ \mathsf{if}\ G_1, \ldots, F\ \mathsf{if}\ G_n \equiv_{\mathcal{C}+} F\ \mathsf{if}\ G_1 \vee \cdots \vee G_n$$
$$(ii)\ F\ \mathsf{if}\ G_1\ \mathsf{after}\ H, \ldots, F\ \ \mathsf{if}\ G_n\ \mathsf{after}\ H \equiv_{\mathcal{C}+} F\ \mathsf{if}\ G_1 \vee \cdots \vee G_n\ \mathsf{after}\ H$$
$$(iii)\ F\ \mathsf{if}\ G\ \mathsf{after}\ H_1, \ldots, F\ \mathsf{if}\ G\ \mathsf{after}\ H_n \equiv_{\mathcal{C}+} F\ \mathsf{if}\ G\ \mathsf{after}\ H_1 \vee \cdots \vee H_n$$

Proof: Using [DIL] and [cDIL]. □

Proposition 42

$$(i) \quad \frac{\mathsf{caused}\ F\ \mathsf{if}\ G}{\mathsf{caused}\ F\ \mathsf{if}\ G \wedge G'} \qquad\qquad (ii) \quad \frac{\mathsf{caused}\ F\ \mathsf{if}\ G\ \mathsf{after}\ H}{\mathsf{caused}\ F\ \mathsf{if}\ G \wedge G'\ \mathsf{after}\ H \wedge H'}$$

Proof: Use [Aug]. Note that either G' or H' may be $\top$. □

Proposition 43 *If* $G \models_\sigma G_1 \vee \cdots \vee G_n$*, then*

$$(i) \quad \frac{\mathsf{caused}\ F\ \mathsf{if}\ G_1, \ldots, \mathsf{caused}\ F\ \mathsf{if}\ G_n}{\mathsf{caused}\ F\ \mathsf{if}\ G}$$

$$(ii) \quad \frac{\mathsf{caused}\ F\ \mathsf{if}\ G_1\ \mathsf{after}\ H, \ldots, \mathsf{caused}\ F\ \mathsf{if}\ G_n\ \mathsf{after}\ H}{\mathsf{caused}\ F\ \mathsf{if}\ G\ \mathsf{after}\ H}$$

Proof: The derivation of (i) using [RDIL] is entirely straightforward.
For (ii), assume $G \models_\sigma G_1 \vee \cdots \vee G_n$. Then clearly by Proposition 31, we have that for any m and i such that $0 \leqslant i < m$, $G[i+1] \models_{\sigma_m} G_1[i+1] \vee \cdots \vee G_n[i+1]$. Thus also, for similar m and i, $G[i+1] \wedge H[i] \models_{\sigma_m} (G_1[i+1] \vee \cdots \vee G_n[i+1]) \wedge H[i]$, so by propositional logic, $G[i+1] \wedge H[i] \models_{\sigma_m} (G_1[i+1] \wedge H[i]) \vee \cdots \vee (G_n[i+1]) \wedge H[i])$. Using [RDIL], we have that (for all i such that $0 \leqslant i < m$)

$$\frac{F[i+1] \Leftarrow G_1[i+1] \wedge H[i], \ldots, F[i+1] \Leftarrow G_n[i+1] \wedge H[i]}{F[i+1] \Leftarrow G[i+1] \wedge H[i]}$$

Using Corollary 10, and translating back into $\mathcal{C}+$ as usual, we have our result. □

Proposition 44 *If $H \models_\sigma H_1 \vee \cdots \vee H_n$, then*

$$\frac{\mathsf{caused}\ F\ \mathsf{if}\ G\ \mathsf{after}\ H_1, \ldots, \mathsf{caused}\ F\ \mathsf{if}\ G\ \mathsf{after}\ H_n}{\mathsf{caused}\ F\ \mathsf{if}\ G\ \mathsf{after}\ H}$$

Proof: Using [RDIL], in the manner of the proof of Proposition 43. □

Proposition 45

$$(i) \quad \frac{\mathsf{caused}\ F'\ \mathsf{if}\ F, \quad \mathsf{caused}\ F\ \mathsf{if}\ G}{\mathsf{caused}\ F'\ \mathsf{if}\ G}$$

$$(ii) \quad \frac{\mathsf{caused}\ F'\ \mathsf{if}\ F, \quad \mathsf{caused}\ F\ \mathsf{if}\ G\ \mathsf{after}\ H}{\mathsf{caused}\ F'\ \mathsf{if}\ G\ \mathsf{after}\ H}$$

$$(iii) \quad \frac{\mathsf{caused}\ F'\ \mathsf{if}\ F\ \mathsf{after}\ H, \quad \mathsf{caused}\ F\ \mathsf{if}\ G\ \mathsf{after}\ H}{\mathsf{caused}\ F'\ \mathsf{if}\ G\ \mathsf{after}\ H}$$

Proof: Uses [MP] and details of the translation from action descriptions to causal theories. □

Other properties of $\mathcal{C}+$ may be established in similar fashion.

7 Example (Winning the lottery)

We now introduce an extended example, to demonstrate how the logical properties we have proved in preceding sections are useful in deciding between different formulations of the same domain. The example is constructed partly to show how $\mathcal{C}+$ deals with indirect effects of actions (ramifications). It also illustrates some issues in the representation of concurrent actions, actions with defeasible effects, and non-deterministic actions. Naturally it is not possible to illustrate everything with one simple example, but the example is indicative of the issues that are encountered when formulating applications in a language as expressive as $\mathcal{C}+$.

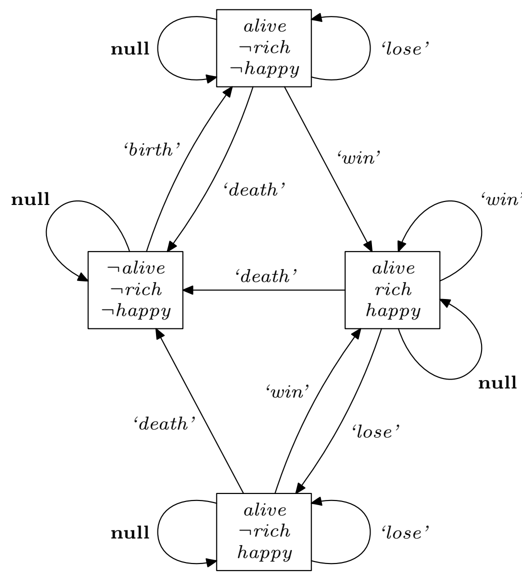
Our example may be summarised in this way: winning the lottery causes one to become (or remain) rich; losing one's wallet causes one to become (or remain) not rich; a person who is rich is happy; a person who is not alive is neither rich nor happy.

The signature has the Boolean simple fluent constants *alive*, *rich*, *happy*, and the Boolean action constants *birth*, *death*, *win*, *lose*:

$$\sigma^f = \{alive, rich, happy\}$$
$$\sigma^a = \{birth, death, win, lose\}$$

The action description and transition system are as follows:

inertial $alive,\ rich,\ happy$
exogenous $birth,\ death,\ win,\ lose$
$birth$ causes $alive$
nonexecutable $birth$ if $alive$
$death$ causes $\neg alive$
nonexecutable $death$ if $\neg alive$
win causes $rich$
nonexecutable win if $\neg alive$
$lose$ causes $\neg rich$
nonexecutable $lose$ if $\neg alive$
caused $happy$ if $rich$
caused $\neg rich$ if $\neg alive$
caused $\neg happy$ if $\neg alive$
nonexecutable $birth \wedge death$
nonexecutable $birth \wedge win$
nonexecutable $birth \wedge lose$
nonexecutable $win \wedge lose$



States and transition labels/events are interpretations of the fluent constants and action constants, respectively. Here, each state and each transition label/event is represented by the set of atoms that it satisfies. Because of the static laws, there are only four states in the transition system and not $2^3 = 8$. The diagram label '*birth*' is shorthand for the label/event $\{birth, \neg death, \neg win, \neg lose\}$, and likewise for the labels '*death*', '*win*' and '*lose*'. The label **null** is shorthand for $\{\neg birth, \neg death, \neg win, \neg lose\}$. The diagram does not show transitions of type $death \wedge lose$, $win \wedge death$, and so on. We will discuss those presently.

Notice that *happy* is declared inertial, and so still persists even if one becomes not rich. That is why the '*lose*' transition from state $\{alive, rich, happy\}$ results in the state $\{alive, \neg rich, happy\}$. We could of course modify the action description so that *happy* is no longer inertial but defined to be true if and only if *rich* is true. Or we might prefer to make *happy* non-inertial and let the '*lose*' transition be non-deterministic. The interactions between these various adjustments are rather subtle, however, and are not always immediately obvious.

We will restrict attention to the following two questions. First, there are alternative ways of formulating the constraints that a person cannot be rich or happy when not alive, and these alternatives have different interactions with the other causal laws. Second, as it turns out, the last group of four nonexecutable statements are all redundant, in that they are already implied by the other causal laws. There are some remaining questions about the effects of concurrent actions in the example which we will seek to identify.

First, let us look at some effects of individual actions. With the static constraints as formulated above, we have the following implied laws. (Henceforth we omit the keyword caused to conserve space.) $death$ causes $\neg alive$ (in other words, $\neg alive$ if $\top$ after $death$) together with $\neg rich$ if $\neg alive$ imply $death$ causes $\neg rich$.

And in general

$$\frac{\alpha \textsf{ causes } F \textsf{ if } G, \quad F' \textsf{ if } F}{\alpha \textsf{ causes } F' \textsf{ if } G}$$

as is easily checked. By a similar argument we also have the implied causal law death causes $\neg \mathit{happy}$ ($\neg \mathit{happy}$ if $\top$ after death) and win causes happy. We do not get the law lose causes $\neg \mathit{happy}$ because as formulated here, we do not have the static law (explicit or implied) $\neg \mathit{happy}$ if $\neg \mathit{rich}$.

Suppose that instead of the static laws $\neg \mathit{rich}$ if $\neg \mathit{alive}$ and $\neg \mathit{happy}$ if $\neg \mathit{alive}$, we had included only the weaker constraints $\bot$ if $\mathit{rich} \wedge \neg \mathit{alive}$ and $\bot$ if $\mathit{happy} \wedge \neg \mathit{alive}$. These constraints eliminate the unwanted states, but are too weak to give the implied effects (ramifications). We also lose transitions: if $\neg \mathit{happy}$ if $\neg \mathit{alive}$ is replaced by either of alive if happy or $\bot$ if $\mathit{happy} \wedge \neg \mathit{alive}$, the only way that $\neg \mathit{happy}$ can be 'caused' is by inertia. Consequently, we eliminate all the death transitions from states in which happy holds: we get the implied law nonexecutable death if happy. (We omit the formal derivation of this implied law for lack of space. It is rather involved since it also requires to taking into account the presence of other causal laws in the example.) Similarly, if we replace $\neg \mathit{rich}$ if $\neg \mathit{alive}$ by either of alive if rich or $\bot$ if $\mathit{rich} \wedge \neg \mathit{alive}$, the only way that $\neg \mathit{rich}$ can be 'caused' is by a lose transition or by inertia. Consequently, transitions of type $\mathit{death} \wedge \neg \mathit{lose}$ become non-executable in the states $\{\mathit{alive}, \mathit{rich}, \neg \mathit{happy}\}$ and $\{\mathit{alive}, \mathit{rich}, \mathit{happy}\}$ whether or not we also make the earlier adjustment to the $\mathit{alive}/\mathit{happy}$ constraint. In addition, we have the implied law nonexecutable $\mathit{death} \wedge \neg \mathit{lose}$ if rich: a rich person cannot die unless he simultaneously loses his wallet.

There is one way in which we can use constraints $\bot$ if $\mathit{rich} \wedge \neg \mathit{alive}$ and $\bot$ if $\mathit{happy} \wedge \neg \mathit{alive}$ (or alive if rich and alive if happy) without losing transitions. That is by adding a pair of extra fluent dynamic laws: either

$$\mathit{death} \textsf{ causes } \neg \mathit{rich} \qquad \text{and} \qquad \mathit{death} \textsf{ causes } \neg \mathit{happy}$$

or the weaker pair death may cause $\neg \mathit{rich}$ and death may cause $\neg \mathit{happy}$. (In $\mathcal{C}+$, α may cause F is an abbreviation for the fluent dynamic law F if F after α.) We leave out the (straightforward) derivation that demonstrates both these pairs have the claimed effect. Neither is entirely satisfactory since they require all ramifications of death to be identified in advance and then modelled explictly using causal laws.

We turn now to examine the effects of concurrent actions. First, notice that the law nonexecutable $\mathit{birth} \wedge \mathit{death}$ is implied by the other causal laws. Because: alive if $\top$ after birth and $\neg \mathit{alive}$ if $\top$ after death imply by [Aug] alive if $\top$ after $\mathit{birth} \wedge \mathit{death}$ and $\neg \mathit{alive}$ if $\top$ after $\mathit{birth} \wedge \mathit{death}$, which in turn together imply by [CC] $\mathit{alive} \wedge \neg \mathit{alive}$ if $\top$ after $\mathit{birth} \wedge \mathit{death}$ (which is equivalent to nonexecutable $\mathit{birth} \wedge \mathit{death}$). And in general

$$\frac{\alpha \textsf{ causes } A \textsf{ if } F, \quad \beta \textsf{ causes } B \textsf{ if } G, \quad C \textsf{ if } A \wedge B}{\alpha \wedge \beta \textsf{ causes } C \textsf{ if } F \wedge G}$$

There is another derivation of nonexecutable $\mathit{birth} \wedge \mathit{death}$ from the causal laws of the example. We have the causal laws nonexecutable birth if alive and nonexecutable death if $\neg \mathit{alive}$. $\bot$ if $\top$ after $\mathit{birth} \wedge \mathit{alive}$ and $\bot$ if $\top$ after $\mathit{death} \wedge \neg \mathit{alive}$ imply by [Aug]: $\bot$ if $\top$ after $\mathit{birth} \wedge \mathit{death} \wedge \mathit{alive}$ and $\bot$ if $\top$ after $\mathit{birth} \wedge \mathit{death} \wedge \neg \mathit{alive}$, which in turn together imply by [DIL] $\bot$ if $\top$ after ($\mathit{birth} \wedge \mathit{death} \wedge$

$alive) \lor (birth \land death \land \neg alive)$, whose antecedent can be simplified by [RCEA]: $\bot$ if $\top$ after $birth \land death$.

In general we have:

$$\frac{\textsf{nonexecutable } \alpha \textsf{ if } F, \quad \textsf{nonexecutable } \beta \textsf{ if } G}{\textsf{nonexecutable } \alpha \land \beta \ \textsf{ if } (F \lor G)}$$

What of $birth \land win$ and $birth \land lose$? We have

$$\frac{\textsf{nonexecutable } birth \textsf{ if } alive, \quad \textsf{nonexecutable } win \textsf{ if } \neg alive}{\textsf{nonexecutable } birth \land win}$$

from which nonexecutable $birth \land lose$ follows by a similar argument.

This leaves transitions of type $death \land lose$ and $death \land win$. $death \land lose$ is not problematic. We have the implied causal laws $death \land lose$ causes $\neg alive$ (by [Aug] from $death$ causes $\neg alive$) and $death \land lose$ causes $\neg rich$ (either by [Aug] from $lose$ causes $\neg rich$ or from the implied law $death$ causes $\neg rich$). In this example, the effects of $death \land lose$ transitions are the same as those of $death \land \neg lose$ transitions.

Consider now $death \land win$. Here we need some adjustment to the example's formulation. We have the implied law nonexecutable $win \land death$ because (one of several possible derivations): we have the implied law $(win \land death)$ causes $(rich \land \neg alive)$, the static law $\neg rich$ if $\neg alive$ implies $\bot$ if $rich \land \neg alive$, and so $(win \land death)$ causes $\bot$, which is equivalent to nonexecutable $win \land death$.

But it seems unreasonable to insist that $win \land death$ transitions cannot occur—that was not the intention when the example was originally formulated. We can admit the possibility of $win \land death$ transitions by re-formulating the relevant causes statement for win so that it reads instead win causes $rich$ if $\neg death$, or equivalently $win \land \neg death$ causes $rich$. The effects of the 'win' transitions are unchanged, but the transition system now contains transitions of type $win \land death$: their effects are exactly the same as those of '$death$' and $death \land lose$ transitions.

But note that after this adjustment, we have to re-examine other combinations of possible concurrent actions. $win \land birth$ is still non-executable (it depended on the pre-conditions of the two actions, not their effects) but we no longer have nonexecutable $win \land lose$. We have only the implied law $(win \land lose)$ causes $(rich \land \neg rich)$ if $\neg death$, or equivalently, nonexecutable $win \land lose \ \land \neg death$. So now a person can win the lottery and lose his wallet simultaneously, but only if he dies at the same time.

But suppose $win \land lose \land \neg death$ *is* intended to be executable. What should its effects be? One possibility is that the effects of win override those of $lose$. We replace the $lose$ causes $\neg rich$ law by the weaker $lose$ causes $\neg rich$ if $\neg win$. A second possibility is that the effects of $lose$ override those of win. We replace the win causes $rich$ if $\neg death$ law by the weaker win causes $rich$ if $\neg death \land \neg lose$. (And we may prefer to introduce an 'abnormality' action constant (see [GLL+04, Section 4.3]) to express the defeasibility of winning more concisely.) The third possibility is to say that $win \land lose$ transitions are non-deterministic:

$$win \land lose \textsf{ may cause } rich, \qquad win \land lose \textsf{ may cause } \neg rich$$

What of the interactions between non-deterministic $win \land lose$ actions and $death$? We still have the implied law $win \land lose \land death$ causes $\neg rich$. But perhaps the non-deterministic effects of the other $win \land lose$ transitions should have been formulated thus:

$$win \land lose \land \neg death \textsf{ may cause } rich, \qquad win \land lose \land \neg death \textsf{ may cause } \neg rich$$

This is unnecessary. In $\mathcal{C}+$ $\{\alpha$ may cause $F,\ \alpha$ may cause $\neg F, \beta$ causes $\neg F\}$ and $\{\alpha \wedge \neg\beta$ may cause $F,\ \alpha \wedge \neg\beta$ may cause $\neg F,\ \beta$ causes $\neg F\}$ are equivalent. Left-to-right is just an instance of [Aug]. For right-to-left, notice that $\alpha\wedge\neg\beta$ may cause F, β causes $\neg F$ is an instance of the general pattern of causal rules $\{P \Leftarrow P \wedge Q \wedge \neg R,\ \neg P \Leftarrow R\}$, discussed at the end of Section 4. It is equivalent to $\{P \Leftarrow P \wedge Q,\ \neg P \Leftarrow R\}$. For the other part, notice that β causes $\neg F$ implies $\alpha \wedge \beta$ may cause $\neg F$ by [Aug], and $\alpha \wedge \neg\beta$ may cause $\neg F$ and $\alpha \wedge \beta$ may cause $\neg F$ together imply α may cause $\neg F$ by [DIL] and [RCEA].

There are other variations of the example that we might consider. We might remove the declaration that *happy* is inertial. Or we might choose to make the fluent constant *happy* statically determined instead of 'simple'. These changes have a further set of interactions with the other causal laws. Their effects can be analysed in similar fashion.

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